J. Fluid Mech. (2000), *vol.* 423, *pp.* 275–291. Printed in the United Kingdom

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Theory of water waves derived from a Lagrangian. Part 1. Standing waves

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(Received 18 April 2000)

A new system of equations for calculating time-dependent motions of deep-water gravity waves (Balk 1996) is here developed analytically and set in a form suitable for practical applications. The method is fully nonlinear, and has the advantage of essential simplicity. Both the potential and the kinetic energy involve polynomial expressions of low degree in the Fourier coefficients $Y_n(t)$. This leads to equations of motion of correspondingly low degree. Moreover the constants in the equations are very simple. In this paper the equations of motion are specialized to standing waves, where the coefficients Y_n are all real. Truncation of the series at low values of $|n|$, say $n < N$, leads to 'partial waves' with solutions apparently periodic in the time t. For physical applications N must however be large. The method will be applied to the breaking of standing waves by the forming of sharp corners at the crests, and the generation of vertical jets rising from the wave troughs.

1. Introduction

The equations of motion for the irrotational motion of an ideal, incompressible fluid with a free surface are commonly derived from a Hamiltonian expression, in which the surface displacement and the velocity potential at the surface are taken as canonical variables (Zakharov 1968). The evaluation of the potential function generally requires a Hilbert transformation, and especially in high-order expansions in powers of a parameter the coefficients in these expansions become extremely complicated; see for example Glozman, Agnon & Stiassnie (1993). A very much simpler and more natural system of equations has been introduced by Balk (1996), based on a Lagrangian. In this system not only are the kinetic and potential energies polynomial expressions of finite degree ($\leqslant 4$) in the independent coordinates (as opposed to infinite order in the case of a Hamiltonian system) but the coefficients in these expressions are mainly low integers. This makes the equations of motion relatively simple to program for numerical computation, and facilitates the discussion of various approximations.

The Lagrangian scheme introduced by Balk for any general, time-dependent deformation of the free surface (including overturning waves) is in fact a generalization of the cubic system of equations for steady, progressive Stokes waves discovered by the present author (Longuet-Higgins 1978). These equations also were found to be derivable from a low-order polynomial Lagrangian (see Longuet-Higgins 1985).

However, the comparative simplicity of the general Lagrangian system has so far remained unexploited. The purpose of the present sequence of papers is to show how Balk's (1996) analysis can be developed, and to examine some of its applications. We begin in the present paper with the case of standing waves, periodic in space but not necessarily in time. General equations are derived in §§ 2 to 4. An important part in the analysis is played by the determinant Δ of the equations of motion, which must not vanish in order for time-stepping to proceed. It is shown in § 5 that this determinant factorizes. Some insights can be gained by considering low-order approximations (i.e. truncations of the Fourier series) as is shown in §§ 6 to 8. High-order approximation will be used in later papers for a discussion of two phenomena displayed by standing waves, namely breaking at the wave crests and the forming of strong vertical jets ('flip-through') rising out of the wave trough.

2. The Lagrangian

It is shown by Balk (1996, § 2) that in any two-dimensional irrotational wave motion, periodic in the (horizontal) x-direction with period 2π, the coordinates (X, Y) of a point on the free surface may be expressed in the form

$$X + \mathrm{i}\,Y = \xi + \sum_{-\infty}^{\infty}(X_n + \mathrm{i}\,Y_n)\,\mathrm{e}^{-\mathrm{i}n\xi}, \tag{2.1}$$

where $X_{-n} = X_n^*$ and $Y_{-n} = Y_n^*$, a star denoting the complex conjugate; also

$$X_n = \mathrm{i}\,\sigma_n Y_n, \tag{2.2}$$

where σ_n equals $+1, -1$ or 0 as n is positive, negative or zero respectively. From (2.1) it follows that

$$\left.\begin{aligned} X &= \xi + \sum_{-\infty}^{\infty} \mathrm{i}\,\sigma_n Y_n\,\mathrm{e}^{-\mathrm{i}n\xi}, \\ Y &= \sum_{-\infty}^{\infty} Y_n\,\mathrm{e}^{-\mathrm{i}n\xi}, \end{aligned}\right\} \tag{2.3}$$

so that the $Y_n(t)$ may be taken as the time-dependent coordinates of the fluid. These are subject to the constraint that the mean level $\overline{Y}$, given by

$$\overline{Y} = \frac{1}{2\pi}\int_0^{2\pi} Y\,\mathrm{d}X = Y_0 + \sum_{m+n=0} n\,\sigma_n\,Y_m Y_n, \tag{2.4}$$

shall vanish. Thus we have always

$$Y_0 = -\sum_{-\infty}^{\infty} |n|\,Y_n\,Y_{-n}. \tag{2.5}$$

The Lagrangian $\mathscr{L}$ is defined as

$$\mathscr{L} \equiv T - V, \tag{2.6}$$

where V and T are the mean potential and kinetic energy densities per unit horizontal distance. As in (2.4), V is found directly from (2.3):

$$2V = \int_0^{2\pi} Y^2\,\mathrm{d}X = \sum_{l+m=0} Y_l\,Y_m + \sum_{l+m+n=0} n\,\sigma_n\,Y_l\,Y_m\,Y_n, \tag{2.7}$$

all the summations running from $-\infty$ to ∞.

In a similar way, the velocity potential $\phi(\xi, t)$ and the streamfunction $\psi(\xi, t)$ at the free surface can be expressed in Fourier series analogous to (2.3):

$$\left.\begin{aligned} \phi &= \sum_{-\infty}^{\infty} \mathrm{i}\,\sigma_n\, B_n\, \mathrm{e}^{-\mathrm{i}n\xi}, \\ \psi &= \sum_{-\infty}^{\infty} B_n\, \mathrm{e}^{-\mathrm{i}n\xi}. \end{aligned}\right\} \tag{2.8}$$

Then the kinetic energy density T is given by

$$2T = \frac{1}{2\pi}\int_0^{2\pi} \phi\, \mathrm{d}\psi = \sum_{-\infty}^{\infty} |n| B_n B_{-n}. \tag{2.9}$$

The crucial step taken by Balk (1996) was to show that the kinematic boundary condition at the free surface can be expressed as

$$\psi_\xi = X_t Y_\xi - Y_t X_\xi, \tag{2.10}$$

which allows the coefficients B_n to be related directly to the Y_n by

$$\mathrm{i}\, n\, B_n = \dot{Y}_n + \sum_{l+m=n} (\sigma_l - \sigma_m) l Y_l \dot{Y}_m \quad (n \neq 0); \tag{2.11}$$

a dot denotes $\mathrm{d}/\mathrm{d}t$. We can always choose

$$B_0 = 0. \tag{2.12}$$

It is convenient to write

$$a_n = \begin{cases} 2|n| Y_n, & n \neq 0 \\ 1, & n = 0, \end{cases} \tag{2.13}$$

so that when, for example, $n < 0$ (2.11) gives

$$\mathrm{i}\, n\, B_n = (\dots a_2 \dot{Y}_{n-2} + a_1 \dot{Y}_{n-1} + a_0 \dot{Y}_n) + \tfrac{1}{2} a_n \dot{Y}_0 + (a_{n-1}\dot{Y}_1 + a_{n-2}\dot{Y}_2 + \cdots). \tag{2.14}$$

From (2.11) we have also

$$\dot{Y}_0 = -(\dots a_4 \dot{Y}_{-4} + a_3 \dot{Y}_{-3} + a_2 \dot{Y}_{-2} + a_1 \dot{Y}_{-1}) - (a_{-1}\dot{Y}_1 + a_{-2}\dot{Y}_2 + \cdots). \tag{2.15}$$

3. Standing waves

So far the equations are quite general. However, in standing waves the coordinates $Y_n(t)$ may be chosen to be all real. The above formulae are then simplified and, from equations (2.8) to (2.13), we find

$$\begin{aligned} 4T =\ & \tfrac{1}{1}[\tfrac{1}{1}(1 + a_2 - a_1 a_1)\dot{a}_1 + \tfrac{1}{2}(a_1 + a_3 - a_1 a_2)\dot{a}_2 + \tfrac{1}{3}(a_2 + a_4 - a_1 a_3)a_3 + \cdots]^2 \\ & + \tfrac{1}{2}[\tfrac{1}{1}(\quad a_3 - a_2 a_1)\dot{a}_1 + \tfrac{1}{2}(1 \ + a_4 - a_2 a_2)\dot{a}_2 + \tfrac{1}{3}(a_1 + a_5 - a_2 a_3)\dot{a}_3 + \cdots]^2 \\ & + \tfrac{1}{3}[\tfrac{1}{1}(\quad a_4 - a_3 a_1)\dot{a}_1 + \tfrac{1}{2}(\qquad a_5 - a_3 a_2)\dot{a}_2 + \tfrac{1}{3}(1 \ + a_6 - a_3 a_3)\dot{a}_3 + \cdots]^2 \\ & + \cdots, \end{aligned} \tag{3.1}$$

or more compactly

$$4T = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} P_{mn}\, \dot{a}_n \right)^2, \tag{3.2}$$

where

$$P_{mn} = \frac{1}{m^{1/2}n}(a_{n-m} + a_{n+m} - a_m\, a_n) \tag{3.3}$$

provided that a_{n-m} is replaced by 0 when $n < m$. Writing equation (3.2) in the form

$$4T = \sum_{m=1}^{\infty}\left(\sum_{k=1}^{\infty} P_{mk}\,\dot{a}_k\right)\left(\sum_{l=1}^{\infty} P_{lk}\,\dot{a}_l\right) \tag{3.4}$$

and reversing the order of the summations we obtain

$$2T = \sum_{1}^{\infty}\sum_{1}^{\infty} Q_{kl}\,\dot{a}_k\,\dot{a}_l, \tag{3.5}$$

where

$$Q_{kl} = \frac{1}{2}\sum_{m=1}^{\infty} P_{mk}\,P_{ml}. \tag{3.6}$$

In other words, the matrix of the quadratic form (3.5) is the column-by-column product of the matrices (P_{mk}) and (P_{ml}). Note that P_{mk} is of maximum degree 2 in the coefficients a_n. Hence T is of degree 2 in $\dot{a}_1, \dot{a}_2, \ldots$ and of maximum degree 4 in $a_1, a_2, \ldots$.

Consider now the degree of the terms in the potential energy V, given by equation (2.7). The right-hand side of (2.7) still contains the dependent variable Y_0. The first summation contains term Y_0^2, while the second contains some terms with no more than one of l, m, n equal to 0 (for, if any two of l, m, n are zero, then so are all three, making $|l|$ vanish). Since by (2.5) Y_0 is quadratic in the remaining Y_n, altogether V has maximum degree 4 in $Y_1, Y_2, \ldots$.

4. The equations of motion

As independent coordinates we may take the coefficients $a_1, a_2, \ldots$ defined by equation (2.13). Lagrange's equations of motion are then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}}{\partial \dot{a}_n}\right) = \frac{\partial \mathscr{L}}{\partial a_n}, \quad n = 1, 2, \ldots, \tag{4.1}$$

which can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{a}_n}\right) - \frac{\partial T}{\partial a_n} = -\frac{\partial V}{\partial a_n}. \tag{4.2}$$

Consider first the terms involving T. From equation (3.5) we have

$$\frac{\partial T}{\partial \dot{a}_n} = \sum_{l=1}^{\infty} Q_{nl}\,\dot{a}_l \tag{4.3}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{a}_n}\right) = \sum_{l} Q_{nl}\,\ddot{a}_l + R_n, \tag{4.4}$$

where

$$R_n = \sum_{l} \frac{\mathrm{d}Q_{nl}}{\mathrm{d}t}\dot{a}_l = \sum_{l}\sum_{k} \frac{\partial Q_{nl}}{\partial a_k}\dot{a}_k\,\dot{a}_l. \tag{4.5}$$

Also on the left of (4.2) we have from equation (3.5)

$$\frac{\partial T}{\partial a_n} = \frac{1}{2} \sum_l \sum_k \frac{\partial Q_{kl}}{\partial a_n} \dot{a}_k \, \dot{a}_l. \tag{4.6}$$

After combining the two quadratic forms (4.5) and (4.6) and writing the latter more symmetrically, the left-hand side of equation (4.2) reduces to

$$\sum_l Q_{nl} \, \ddot{a}_n - \sum_l \sum_k S_n(l, k) \dot{a}_k \, \dot{a}_l, \tag{4.7}$$

where

$$S_n(k, l) = \frac{1}{2} \left(\frac{\partial Q_{kl}}{\partial a_n} - \frac{\partial Q_{nl}}{\partial a_k} - \frac{\partial Q_{nk}}{\partial a_l} \right). \tag{4.8}$$

Note that when $k = n$ or $l = n$, two of the terms in this expression cancel.

In order to calculate $\partial Q_{kl}/\partial a_n$ we have from (3.6)

$$\frac{\partial Q_{kl}}{\partial a_n} = \frac{1}{2} \sum_m \left(P_{mk} \frac{\partial P_{ml}}{\partial a_n} + P_{ml} \frac{\partial P_{mk}}{\partial a_n} \right) \tag{4.9}$$

while from (3.3)

$$\begin{aligned} \frac{\partial P_{ml}}{\partial a_n} = 0 & \\ & +1 \quad \text{if } n = l - m \\ & +1 \quad \text{if } n = l + m \\ & -a_m \quad \text{if } n = l \\ & -a_l \quad \text{if } n = m. \end{aligned} \tag{4.10}$$

Lastly for the terms involving V in equations (4.2) and (4.7), note that

$$\frac{\partial V}{\partial a_n} = \sum_{k=-\infty}^{\infty} \frac{\partial V}{\partial Y_k} \frac{\partial Y_k}{\partial a_n}, \tag{4.11}$$

where

$$\frac{\partial Y_k}{\partial a_n} = \begin{cases} \dfrac{1}{2|n|} & \text{if } k = \pm n \\ -Y_k & \text{if } k = 0 \\ 0 & \text{otherwise;} \end{cases} \tag{4.12}$$

also that

$$\frac{\partial}{\partial Y_k}(|l| \; Y_l Y_m Y_n) = \begin{cases} |l| \; Y_m Y_n & \text{if } k = l \\ |l| \; Y_l Y_n & \text{if } k = m \\ |l| \; Y_l Y_m & \text{if } k = n. \end{cases} \tag{4.13}$$

To summarize, Lagrange's equations of motion (4.1) can be expressed in the form

$$\sum_l Q_{nl} \, \ddot{a}_l = \sum_l \sum_k S_n(k, l) \dot{a}_k \, \dot{a}_l + U_n, \quad n = 1, 2 \ldots, \tag{4.14}$$

where Q_{nl} is of maximum degree 4 in the coordinates $a_1, a_2, \ldots$, while S_n and $U_n = -\partial V/\partial a_n$ are each of maximum degree 3.

5. Solution of the equations

In order to solve equations (4.14) we may suppose $a_n(t) \equiv 0$ for all n greater than N, say, and proceed by successive approximations as N is increased indefinitely. Given some starting values for a_n and $\dot{a}_n$ at an initial time $t = 0$, equations (4.13) comprise a set of simultaneous equations for determining $\ddot{a}_1, \ddot{a}_2, \ldots \ddot{a}_N$. These can be solved provided that the determinant

$$\Delta_N = \|Q_{ij}\| \tag{5.1}$$

does not vanish. Some interest therefore attaches to the conditions under which

$$\Delta_N = 0. \tag{5.2}$$

Now from equation (3.6) we have

$$\Delta_N = \frac{1}{2^N}\|P_{ij}\|^2 \tag{5.3}$$

so that the vanishing of Δ_N depends solely on the vanishing of $\|P_{ij}\|$. Moreover from equation (3.3) we have

$$\|P_{ij}\| = \frac{1}{(N!)^{3/2}} D_N, \tag{5.4}$$

where D_N is the $(N \times N)$ determinant

$$D_N = \begin{vmatrix} (1 + a_2 - a_1 a_1) & (a_1 + a_3 - a_1 a_2) & (a_2 + a_4 - a_1 a_3) & \cdots \\ (\quad a_3 - a_2 a_1) & (1 \ + a_4 - a_2 a_2) & (a_1 + a_5 - a_2 a_3) & \cdots \\ (\quad a_4 - a_3 a_1) & (\quad a_5 - a_3 a_2) & (1 \ + a_6 - a_3 a_3) & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}. \tag{5.5}$$

Consider now the $(N+1) \times (N+1)$ determinant

$$E_N = \begin{vmatrix} 1 & a_1 & a_2 & a_3 & \cdots \\ a_1 & (1 + a_2) & (a_1 + a_3) & (a_2 + a_4) & \cdots \\ a_2 & a_3 & (1 + a_4) & (a_1 + a_5) & \cdots \\ a_3 & a_4 & a_5 & (1 + a_6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}. \tag{5.6}$$

On subtracting from the ith column of the determinant a_i times the first column we reduce each element of the first row to 0, except for the first element which is 1. It follows that

$$D_N = E_N. \tag{5.7}$$

Now in E_N the sum of the elements of each column is

$$1 + a_1 + a_2 + \cdots + a_N = \sum\nolimits^{(+)}, \tag{5.8}$$

say. Each element in the first row may therefore be replaced by $\sum^{+}$. Hence $\sum^{+}$ is a factor of the determinant E_N. Now multiplying the second row by 2, adding to this $2\times$ each of the other *even* rows and subtracting from this the first row, we see that all the elements of the second row become equal to $\pm\sum^{(-)}$ where

$$\sum\nolimits^{(-)} = 1 - a_1 + a_2 - a_3 + a_4 - \cdots . \tag{5.9}$$

Hence $\sum^{(-)}$ is also a factor of E_N. After extraction of these two factors the remaining determinant is

$$F_N = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ -1 & 1 & -1 & 1 & \cdots \\ a_2 & a_3 & (1+a_4) & (a_1+a_5) & \cdots \\ a_3 & a_4 & a_5 & (1+a_6) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{vmatrix}. \tag{5.10}$$

Altogether then, from equations (5.3), (5.4) and (5.10) we have

$$\Delta_N = \frac{1}{2^N (N!)^3}\left(\sum\nolimits^{(+)}\sum\nolimits^{(-)} F_N\right)^2 . \tag{5.11}$$

As will be seen below, the vanishing of $\sum^{(+)}$ or $\sum^{(-)}$ is associated with the formation of a cusp at the free surface.

If T_N and V_N denote the truncated forms of the kinetic and potential energies, and if $\mathscr{L}_N = T_N - V_N$ we note that an exact integral of the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathscr{L}_N}{\partial \dot{a}_n}\right) = \frac{\partial \mathscr{L}_N}{\partial a_n}, \quad n = 1, 2, \ldots N, \tag{5.12}$$

is

$$H_N \equiv T_N + V_N = \text{constant}. \tag{5.13}$$

Equation (5.13) may be derived on multiplying each side of (5.12) by $\dot{a}_n$ and then summing with respect to n from 1 to N. For, if f is any function of $a_1, \ldots a_N$ and $\dot{a}_1, \ldots \dot{a}_N$, formally independent of the time t, we have identically

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \sum_1^N \left(\dot{a}_n \frac{\partial f}{\partial a_n} + \ddot{a}_n \frac{\partial f}{\partial \dot{a}_n}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_1^N \dot{a}_n \frac{\partial f}{\partial \dot{a}_n}\right) - \sum_1^N \dot{a}_n \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial f}{\partial \dot{a}_n} - \frac{\partial f}{\partial a_n}\right). \end{aligned} \tag{5.14}$$

When $f \equiv \mathscr{L}_N$ the last group of terms vanishes by (5.12), and since T_N is quadratic in $\dot{a}_1, \ldots \dot{a}_N$ we obtain

$$\frac{\mathrm{d}\mathscr{L}_N}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} 2T_N, \tag{5.15}$$

whence on integration (5.13) follows.

In the following two sections we shall explore the two lowest-order approximations $N = 1$ and $N = 2$. Although their validity is strictly limited to small values of a_1 and a_2, the behaviour of the approximations at larger values of a_1 and a_2, and the manner in which the approximations fail, will be seen to be of interest in later work.

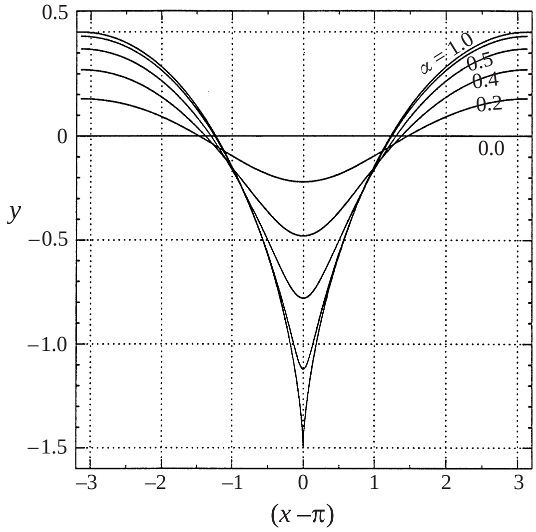

FIGURE 1. Family of surface profiles given by equations (6.1) (the case $N = 1$).

6. The case $N = 1$

In the simplest case we set $a_2 = a_3 = \cdots = 0$, so that the free surface profile is given by

$$\left.\begin{aligned} x &= \xi \ + \alpha \sin \xi, \\ y &= Y_0 + \alpha \cos \xi, \end{aligned}\right\} \tag{6.1}$$

where $\alpha = a_1$ and $Y_0 = -\frac{1}{2}\alpha^2$. This is the parametric equation of a cycloid; see figure 1. Physically admissible solutions are limited to $|\alpha| \leqslant 1$. When $\alpha = \pm 1$ the profile has a downward-pointing cusp.

The dynamical equations take into account only the self-interaction of the fundamental harmonic. Thus from § 2 we have

$$\left.\begin{aligned} 4T_1 &= (1-\alpha^2)^2\dot{\alpha}^2, \\ 4V_1 &= \alpha^2 - \tfrac{1}{2}\alpha^4. \end{aligned}\right\} \tag{6.2}$$

The single Lagrange equation (5.12) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\tfrac{1}{2}(1-\alpha^2)^2\dot{\alpha}\right] = -\alpha(1-\alpha^2)\dot{\alpha}^2 - \tfrac{1}{2}\alpha(1-\alpha^2) \tag{6.3}$$

and after dividing through by $(1-\alpha^2)$ we obtain

$$(1-\alpha^2)\ddot{\alpha} - 2\alpha\dot{\alpha}^2 + \alpha = 0. \tag{6.4}$$

The energy integral (5.13) is

$$(1-\alpha^2)^2\dot{\alpha}^2 + (\alpha^2 - \tfrac{1}{2}\alpha^4) = 4H, \tag{6.5}$$

whence

$$\dot{\alpha}^2 = \frac{4H - \alpha^2(1-\frac{1}{2}\alpha^2)}{(1-\alpha^2)^2} \tag{6.6}$$

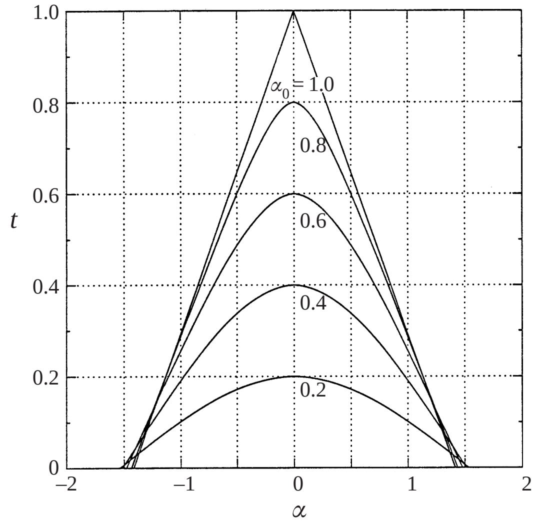

FIGURE 2. Graphs of $\alpha(t)$ covering half a wave period, for different values of $\alpha_0 = \alpha(0)$, assuming that $\dot{\alpha}(0) = 0$.

and

$$t = \int_{\alpha}^{\alpha_0} \frac{(1-\alpha^2)}{[4H - \alpha^2(1-\frac{1}{2}\alpha^2)]^{1/2}}\,\mathrm{d}\alpha. \tag{6.7}$$

Equation (6.7) can also be written as

$$t = \pm\sqrt{2}\int_{\alpha}^{\alpha_0} \frac{1-\alpha^2}{[(1-\alpha^2)^2 - C^2]^{1/2}}\,\mathrm{d}\alpha, \tag{6.8}$$

where

$$C^2 = (1-\alpha_0^2)^2 = 1 - 8H. \tag{6.9}$$

Figure 2 shows α as a function of the time t for various values of $\alpha_0 = \alpha(0)$, assuming that at $t = 0$ the fluid is at rest. When α_0 is small, $\alpha(t)$ is simply a sine-wave with period 2π. Only the section of each curve for which $\alpha > 0$ is shown. In the hypothetical limiting case $\alpha_0 = 1$ we have $C = 0$ and so equation (6.9) gives

$$t = \pm\sqrt{2}(1-\alpha), \tag{6.10}$$

that is to say a pair of straight-line segments. The crest of the wave then rises or falls uniformly when $t \neq 0$, with an instantaneous, large negative acceleration at $t = 0$.

7. The case $N = 2$

On writing $a_1 = \alpha$ and $a_2 = \beta$, the free surface is now given by

$$\left.\begin{aligned} x &= \xi \;\; + \alpha\sin\xi + \tfrac{1}{2}\beta\sin 2\xi, \\ y &= Y_0 + \alpha\cos\xi + \tfrac{1}{2}\beta\cos 2\xi, \end{aligned}\right\} \tag{7.1}$$

where

$$Y_0 = -\tfrac{1}{2}\alpha^2 - \tfrac{1}{4}\beta^2. \tag{7.2}$$

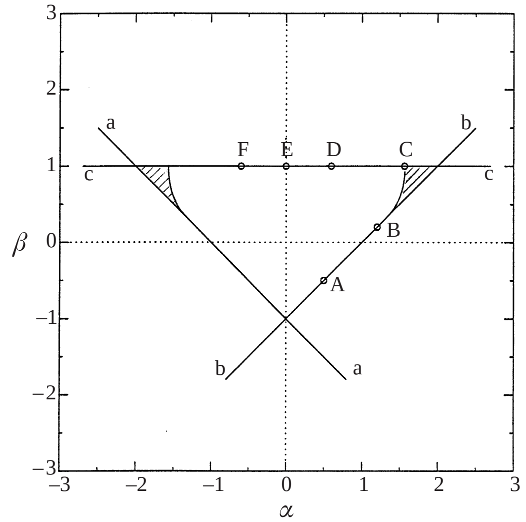

FIGURE 3. Outer limits for the point $(\alpha, \beta) = (a_1, a_2)$ in the (α, β)-plane.

Outer limits for α and β are given by the vanishing of Δ_N in equation (5.11). Since in (5.10)

$$F_3 = \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & -1 \\ \beta & 0 & 1 \end{vmatrix} = (1-\beta), \tag{7.3}$$

the outer limits are given by

$$(1+\alpha+\beta)(1-\alpha+\beta)(1-\beta) = 0 \tag{7.4}$$

which represents the three lines

$$\alpha+\beta = -1, \quad \alpha-\beta = 1 \quad \text{and} \quad \beta = 1. \tag{7.5}$$

These are shown in figure 3, indicated by the symbols aa, bb and cc respectively.

It appears that whenever the point (α, β) lies on any of the lines (7.5) the corresponding profile (7.1) has a downward-pointing cusp. Some examples, indicated by the points A, B, C, D and E in figure 3, are shown in figure 4. In addition, there are limits arising from the need to avoid loops or double points in the profile. These limits correspond to the curved boundaries in figure 3. Thus (α, β) is limited to the interior of the triangle defined by the lines (1), (2) and (3), and excluding the shaded areas in the two upper corners.

The evolution equations are derived from the Lagrangian $\mathscr{L}_2 = T_2 - V_2$ where

$$\left.\begin{aligned} 4T_2 &= [(1+\beta-\alpha^2)\dot{\alpha} + \tfrac{1}{2}(\alpha-\alpha\beta)\dot{\beta}]^2 + \tfrac{1}{2}[\alpha\beta\dot{\alpha} - \tfrac{1}{2}(1-\beta^2)\dot{\beta}]^2, \\ 4V_2 &= (\alpha^2 + \tfrac{1}{4}\beta^2) + \alpha^2\beta - \tfrac{1}{2}(\alpha^2 + \tfrac{1}{2}\beta^2)^2. \end{aligned}\right\} \tag{7.6}$$

The equations of motion for α and β are then

$$\left.\begin{aligned} L\ddot{\alpha} + M\ddot{\beta} &= -\tfrac{1}{2}[L_\alpha\dot{\alpha}^2 + 2L_\beta\dot{\alpha}\dot{\beta} + (2M_\beta - N_\alpha)\dot{\beta}^2] + C_1, \\ M\ddot{\alpha} + N\ddot{\beta} &= -\tfrac{1}{2}[(2M_\alpha - L_\beta)\dot{\alpha}^2 + 2N_\alpha\dot{\alpha}\dot{\beta} + N_\beta\dot{\beta}^2] + C_2, \end{aligned}\right\} \tag{7.7}$$

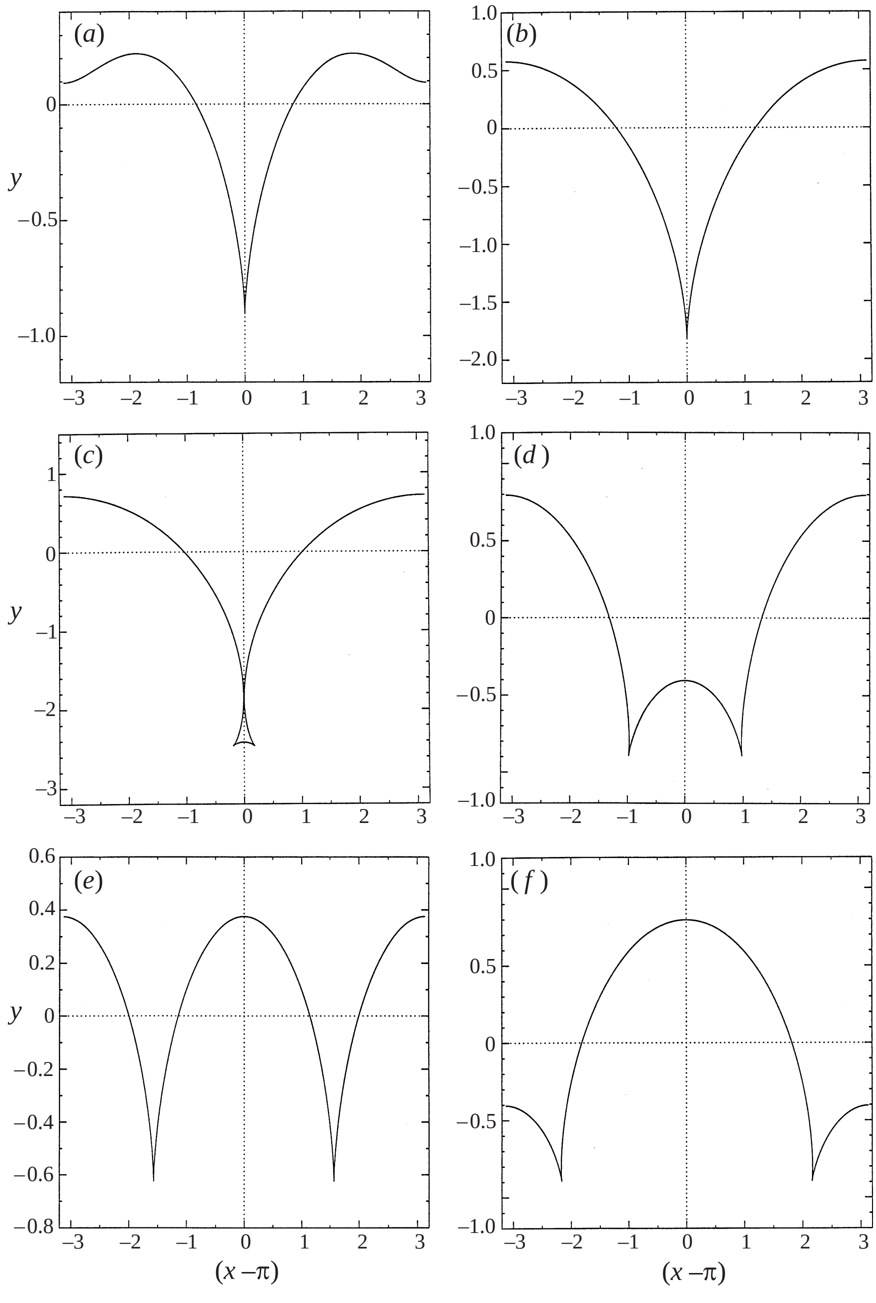

FIGURE 4. (a–f) Surface profiles corresponding to the points A to F in figure 3.

where

$$\left.\begin{aligned} L &= 2(1+\beta-\alpha^2)^2+\alpha^2\beta^2, \\ M &= \alpha(1-\beta)(1+\beta-\alpha^2)-\tfrac{1}{2}\alpha\beta(1-\beta^2), \\ N &= \tfrac{1}{2}\alpha^2(1-\beta)^2+\tfrac{1}{4}(1-\beta^2)^2, \end{aligned}\right\} \tag{7.8}$$

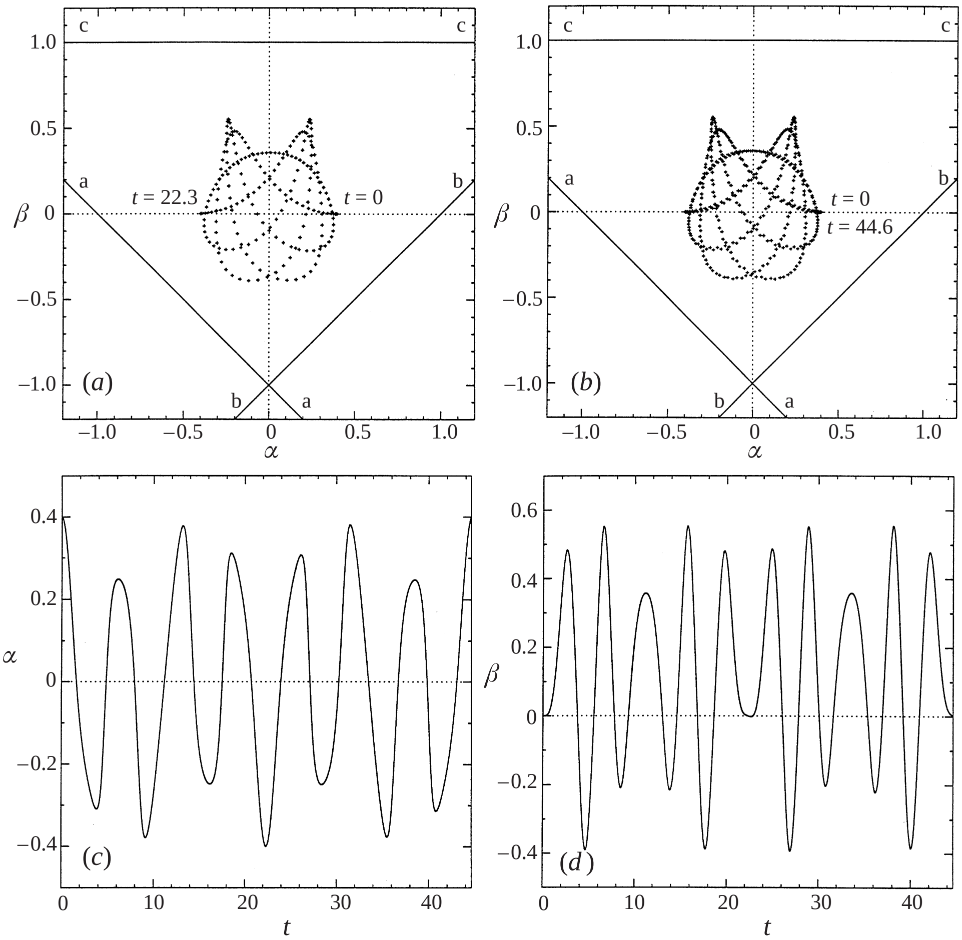

FIGURE 5. (*a*) Trajectory of (α, β) when $N = 2$ and with starting values $\alpha = 0.4$, $\beta = 0$, $\dot{\alpha} = \dot{\beta} = 0$. Time interval $0 \leqslant t \leqslant 22.3$. Interval Δt between plots $= 0.1$. (*b*) As in (*a*) but with $0 \leqslant t \leqslant 44.6$. (*c*) Graph of $\alpha(t)$ for (*a*), showing ten oscillations. (*d*) Graph of $\beta(t)$ for (*b*), showing seven oscillations.

and

$$\left.\begin{aligned} C_1 &= 2\alpha(1 + \beta - \alpha^2) - \alpha\beta^2, \\ C_2 &= \alpha^2(1 - \beta) - \tfrac{1}{2}\beta(1 - \beta^2). \end{aligned}\right\} \tag{7.9}$$

Some examples of the trajectories of (α, β) are shown in figures 5 to 7. In each case the motion starts from rest: $\dot{\alpha}(0) = \dot{\beta}(0) = 0$, so that the total energy H is just equal to the initial potential energy $V(0)$.

For disturbances of very small initial amplitude one would expect the coefficients α and β to oscillate independently, each with its own frequency. Thus α would oscillate harmonically with frequency 1 and β with frequency $\sqrt{2} = 1.414\ldots$. The ratio of the frequencies being irrational, the combined motion would be non-periodic, in the small-amplitude limit. However, very slight nonlinearity may cause the frequencies to become compatible, resulting in a periodic orbit. One such example is shown in figure 5(*a*) where the initial amplitudes are

$$\alpha = -0.4, \qquad \beta = 0. \tag{7.10}$$

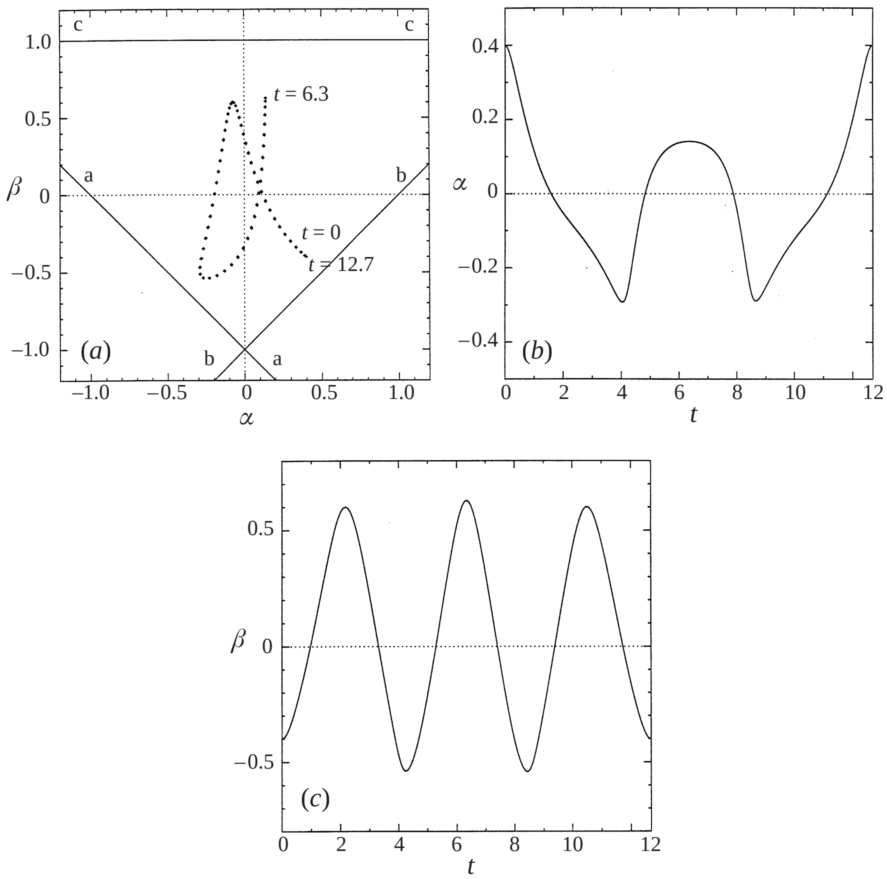

FIGURE 6. (a) Trajectory of (α, β) when $N = 2$ and with starting values $\alpha = 0.4$, $\beta = -0.4$, $\dot{\alpha} = \dot{\beta} = 0$. $t = 0.0(0.1)12.7$. (b) Graph of $\alpha(t)$ for (a). (c) Graph of $\beta(t)$ for (a).

The crosses mark points on the trajectory separated by time-intervals 0.1 (one hundred time steps Δt). After time $t = 22.3$ the point (α, β) is close to the opposite point (0.4, 0.0) where, because of conservation of energy, the velocities α are very small. At time $t = 44.6$, (α, β) has returned practically to its initial position. In the intervening time, α has executed seven complete oscillations and β has executed ten (see figures 5b and 5c). So the ratio of 'frequencies' is $10 \div 7$, i.e. 1.429 approximately.

A simpler example is shown in figure 6(a), where the starting point is

$$\alpha = 0.4, \qquad \beta = -0.4. \tag{7.11}$$

At $t = 12.8, (\alpha, \beta)$ returns to its starting point, α having executed two oscillations and β three, the ratio of frequencies being 1.5. Compared to $\alpha(t)$, the form of $\beta(t)$ is remarkably sinusoidal.

A third example is shown in figure 7(a). Here the initial amplitudes are

$$\alpha = 0.5, \qquad \beta = 0. \tag{7.12}$$

The initial energy is sufficiently large that after just over half an oscillation of the fundamental $\alpha(t)$ the trajectory of (α, β) runs off the scale. The point (α, β) accelerates toward the boundary, as is shown by the curve of $\alpha(t)$ in figure 7(b). The critical

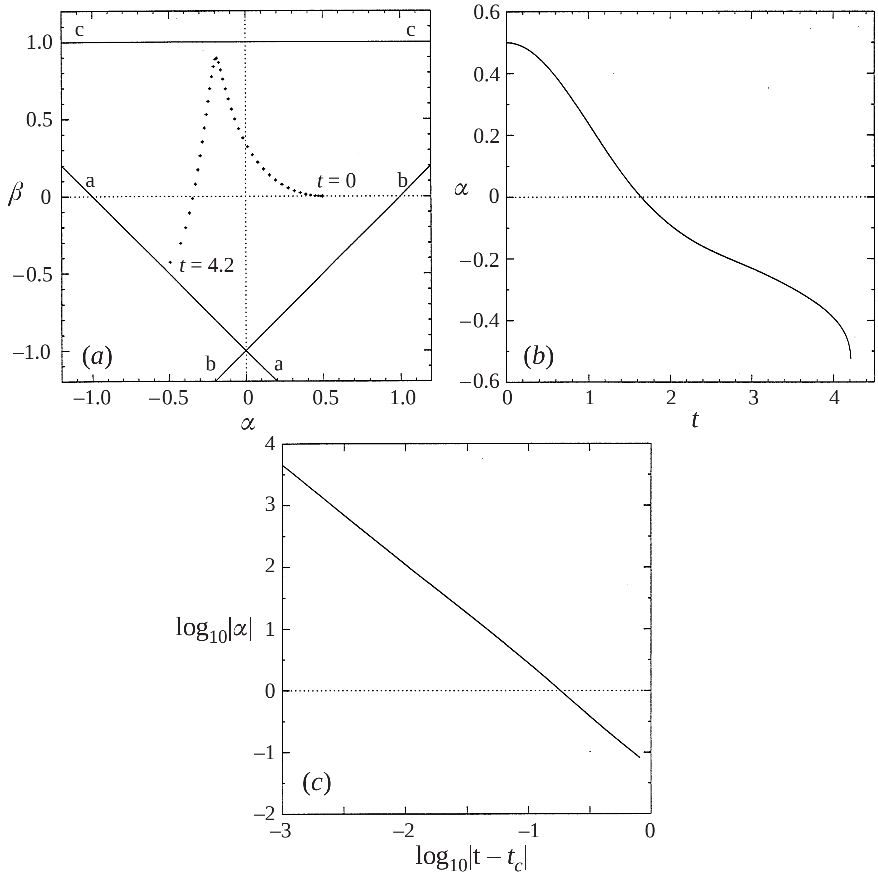

FIGURE 7. (a) Trajectory of (α, β) when $N = 2$ and with starting values $\alpha = 0.5$, $\beta = 0$, $\dot{\alpha} = \dot{\beta} = 0$. ($b$) Graph of $\alpha(t)$ for (a). (c) Logarithmic plot of $\ddot{\alpha}(t)$ near the critical instant $t_c = 4.211$.

time t at which (α, β) crosses the boundary is $t_c = 4.211$ approximately. The final acceleration $|\ddot{\alpha}|$ as t approaches t_c is shown in figure 7(c) on a logarithmic scale. The acceleration varies as $|t - t_c|^{\lambda}$, where λ is a constant. This behaviour is typical of any function $f(t)$ satisfying an equation of the form

$$f\ddot{f} = C\dot{f}^2 \tag{7.13}$$

when f passes through a zero.

In all of the above numerical integrations the total energy H remained a constant to within one part in 10^5.

8. The case $N = 3$

This last special case to be discussed is the simplest for which not all the factors of Δ_N are linear. Thus from equation (5.11) on writing $N = 3$ and $a_1, a_2, a_3 = \alpha, \beta, \gamma$ respectively we have

$$\Delta_3 = \frac{1}{12^3}(1 + \alpha + \beta + \gamma)^2(1 - \alpha + \beta - \gamma)^2 F_4^2, \tag{8.1}$$

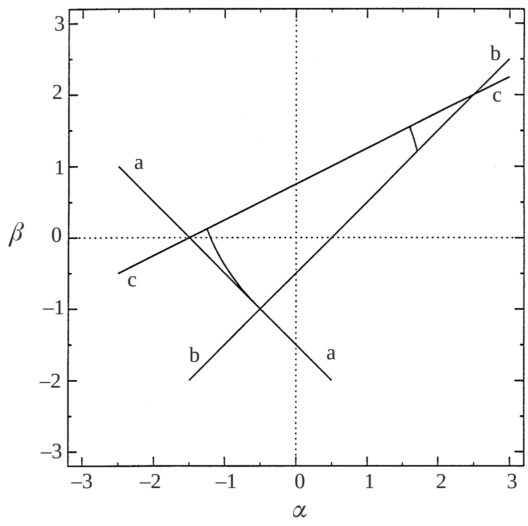

FIGURE 8. Outer limits for $(\alpha, \beta) = (a_1, a_2)$ in the case $N = 3$, when $a_3 = 0.5$.

where by (5.10)

$$F_4 = \begin{vmatrix} 0 & 1 & 0 & 1 \\ -1 & 1 & -1 & 1 \\ \beta & \gamma & 1 & \alpha \\ \gamma & 0 & 0 & 1 \end{vmatrix}, \tag{8.2}$$

which on expansion gives

$$F_3 = (1-\beta) + (\alpha - \gamma)\gamma. \tag{8.3}$$

Whereas in (8.1)

$$1 + \alpha + \beta + \gamma = 0 \tag{8.4}$$

and

$$1 - \alpha + \beta - \gamma = 0 \tag{8.5}$$

represent bounding planes in the space of (α, β, γ), the last factor

$$(1-\beta) + (\alpha - \gamma)\gamma = 0 \tag{8.6}$$

represents a quadric surface (hyperboloid). Nevertheless the intersection of this surface with any plane $\gamma =$ constant is a straight line in that plane. When $\gamma = 0$, of course, equations (8.4), (8.5) and (8.6) reduce to the lines aa, bb and cc of figure 3. In the more typical case when $\gamma = 0.5$, these lines appear shifted as in figure 8. Thus aa is lowered by an amount γ, bb is raised by γ and cc is tilted with inclination γ and passes through the point $(\alpha, \beta) = (\gamma, 1)$. We may call the triangle formed by aa, bb and cc the basic triangle.

The origin $(0, 0)$, which corresponds to infinitesimal orbits, always lies in the interior of the basic triangle so long as $|\gamma| < 1$, and it can be verified that points (α, β) lying on the sides of the basic triangle always correspond to surface profiles having downwards-pointing cusps.

We show one example of a time-history $\alpha(t), \beta(t), \gamma(t)$ when the initial conditions

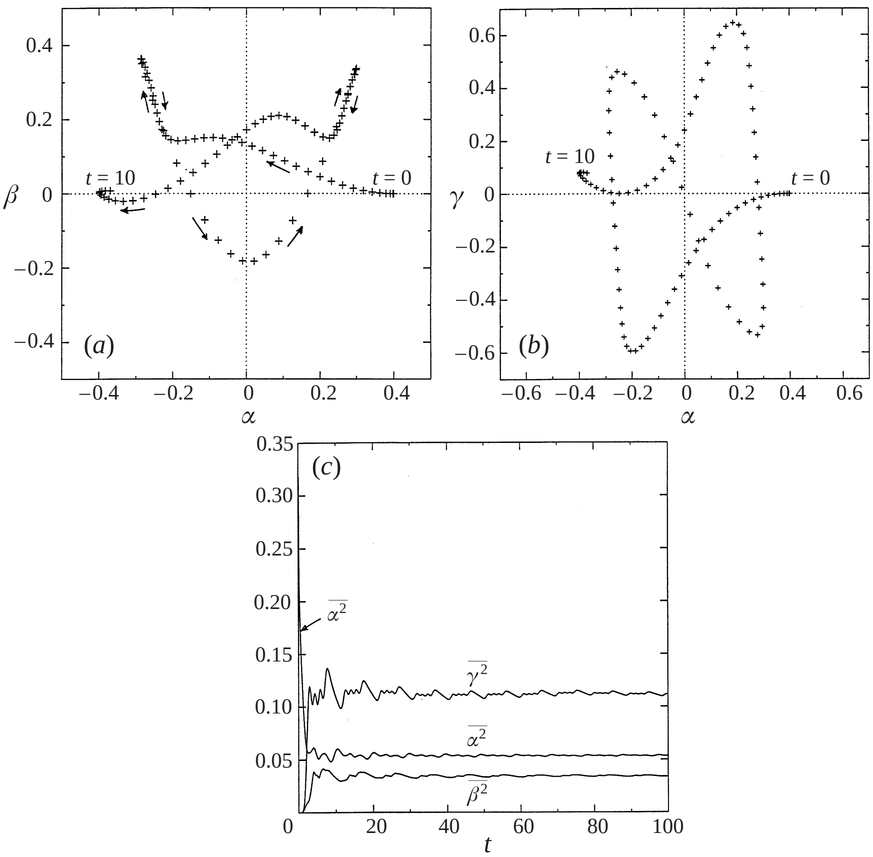

FIGURE 9. (*a*) Trajectory of (α, β) when $N = 3$ and with starting values $\alpha = 0.4$, $\beta = \gamma = 0$, $\dot{\alpha} = \dot{\beta} = \dot{\gamma} = 0$ (compare figure 5*a*). (*b*) Trajectory of (α, γ) corresponding to (*a*). (*c*) Root-mean-square values of α, β and γ as functions of the time t.

are that at time $t = 0$

$$\alpha = 0.4, \qquad \beta = \gamma = 0 \tag{8.7}$$

and $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$ all vanish as before. A plot of β vs. α alone is shown in figure 9(*a*), and the corresponding plot of γ vs. α in figure 9(*b*). Comparing figure 9(*a*) with figure 6(*a*), where the initial conditions were similar except that γ was restricted to be 0, we see that trajectories of (α, β) appear very different and the motion is less periodic. An approximate complete cycle $0 < t < 44.0$, is shown in figure 9(*c*).

It will be noted that the r.m.s. values of α, β and γ are all of comparable magnitude.

9. Conclusions and discussion

We have transformed Balk's (1996) system of equations into a more usable form and specialized them to the case of standing waves. The general system of equations (4.14) for the independent coefficients $a_n(t), n = 1, 2, \ldots$ may be truncated at $n = N$ and used for time-stepping the motion, provided that the determinent Δ_N of the system does not vanish, as is true for sufficiently small a_n; see equation (5.5).

Low-order representations of standing waves have well-defined zones of existence in a_n-space. The boundaries are mostly straight lines or planes. As these boundaries are approached, the corresponding surface profiles are found to develop downward-pointing cusps. In the case $N = 2$ a choice of simple initial conditions strongly suggests the existence of periodic 'waves' analagous to those corresponding to a truncated Hamiltonian which were found by Glozman *et al.* (1993). However, the addition of a non-zero third harmonic ($N = 3$) radically affects the periodicity. The motion then appears to be chaotic, in a way resembling that found in truncated models of progressive waves. However, as noted by Glozman *et al.* (1993) the stochastic properties of truncated models depend strongly on the order of truncation. In order to study chaos or periodicity in 'real' waves it is necessary to take the approximation to high orders, as was done by Agnon & Glozman (1996) for a Hamiltonian system. We do not pursue the subject here.

An application of the equations to some interesting aspects of standing-wave behaviour which require approximations of large order N leading to high numerical accuracy will be given in a future paper (Longuet-Higgins 2000).

This work has been supported by the US Office of Naval Research under Contract N00014-94-1-0008.

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